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On the two-point correlation of potential vorticity in rotating and stratified turbulence

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A framework is developed to describe the two-point statistics of potential vorticity in rotating and stratified turbulence as described by the Boussinesq equations. The Kármán–Howarth equation for the dynamics of the two-point correlation function of potential vorticity reveals the possibility of inertial-range dynamics in certain regimes in the Rossby, Froude, Prandtl and Reynolds number parameters. For the case of large Rossby and Froude numbers, and for the case of quasi-geostrophic dynamics, a linear scaling law with 2/3 prefactor is derived for the third-order mixed correlation between potential vorticity and velocity, a result that is analogous to the Kolmogorov 4/5-law for the third-order velocity structure function in turbulence theory.

1. Introduction

Potential vorticity $q = \boldsymbol{\omega}_a \cdot \nabla \rho$, where $\boldsymbol{\omega}_a$ is the total vorticity and ρ is the density, is a Lagrangian invariant in rotating and stratified flows (Ertel 1942). In the quasi-geostrophic (QG) limit, in which the rotation and stratification effects are strong, the potential vorticity evolution entirely determines the dynamics of quantities such as wind speed, pressure and temperature fields (see for example, Hoskins, McIntyre & Robertson 1985; Rhines 1986; Muller 1995; Haynes & McIntyre 1990). Instead of studying q as a local invariant in the traditional way, we here focus on the global statistical description of the potential vorticity field in the style of the Kolmogorov 1941 theory (K41) for the statistical description of the velocity field (Kolmogorov 1941a, b, c). We treat potential vorticity as a statistical variable and consider the dynamics of its second-order moment in the triply periodic or infinite domain. Thus the potential enstrophy $Q = \langle q^2 \rangle /2$ is the conserved quantity of interest, where $\langle \cdot \rangle$ denotes a suitable average (over ensembles or over the flow domain).

Charney (1971) showed analytically that in the QG limit, conservation of energy and potential enstrophy implies an inverse cascade of energy with energy spectrum scaling as $E(k) \sim k^{-5/3}$ for the low wavenumbers, and a forward cascade of potential enstrophy with $E(k) \sim k^{-3}$ at high wavenumbers. The strong constraint on the energy by Q is analogous to the constraint on energy by enstrophy in two-dimensional turbulence, resulting in the well-known inverse cascade of energy, the forward cascade of enstrophy, and the corresponding scalings of the energy spectrum (Kraichnan 1971). Herring, Kerr & Rotunno (1994) performed moderate-resolution spectral simulations of non-rotating non-stratified turbulent flow with a passive scalar θ, in which the Ertel potential vorticity $q = \boldsymbol{\omega}\cdot\nabla\theta$ with $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$. They studied the evolution of the potential enstrophy spectrum and demonstrated that the transfer terms and the dissipation terms are not separated in wavenumber space. They argued therefore, that an 'inertial' range of scales dominated by the flux of potential enstrophy does not seem possible. The concept of an inertial range is a cornerstone of the K41 theory of turbulence. Therefore it seemed futile, in a certain sense, to further explore statistical approaches for potential vorticity dynamics.

To revisit the possibility of potential enstrophy intertial-range dynamics, we depart from previous approaches in two ways. First, we confine ourselves to statistics in physical space, and undertake a novel study of the evolution equations for the two-point spatial correlation function of potential vorticity. Second, we examine, in addition to the QG and non-rotating, non-stratified limits, four other regimes of interest in the rotation and stratification parameters. Our approach has revealed that some of these regimes do allow for the existence of an inertial range of potential enstrophy. Furthermore, when we take a prescribed sequence of limits in the non-dimensional parameters and assume local isotropy, then exact scaling laws for small-scale potential vorticity statistics emerge. These new laws are analogous to existing laws that form the backbone of statistical hydrodynamics in three-dimensional non-rotating non-stratified flows.

Following K41 turbulence theory, the von Kármán & Howarth (1938) equations for the two-point correlation functions of velocity lead to the concept of an 'inertial range' of scales dominated by net downscale flux of kinetic energy by nonlinear transfer with little dissipative loss. The inertial-range concept was extended to the flux of passive scalar variance by Yaglom (1949), and more recently to helicity flux by Chkhetiani (1996) (see also L'vov, Podivilov & Procaccia 1997; Gomez, Politano & Pouquet 2000; Kurien 2003). Under the assumptions of statistical homogeneity and statistical isotropy of the small scales (local isotropy), all these results take the form of exact scaling laws for the appropriate two-point third-order correlations as follows:

$$\langle (u_L(\boldsymbol{x}+\boldsymbol{r}) - u_L(\boldsymbol{x}))^3 \rangle = -\tfrac{4}{5}\varepsilon r,\ \ \varepsilon = 2\nu\langle \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}\rangle, \qquad (1.1a)$$

$$\langle (u_L(\boldsymbol{x}+\boldsymbol{r}) - u_L(\boldsymbol{x}))(\theta(\boldsymbol{x}+\boldsymbol{r}) - \theta(\boldsymbol{x}))^2 \rangle = -\tfrac{4}{3}\varepsilon_\theta r,\ \ \varepsilon_\theta = 2\alpha\langle \nabla\theta\cdot\nabla\theta\rangle, \qquad (1.1b)$$

$$\langle (u_L(\boldsymbol{x}+\boldsymbol{r}) - u_L(\boldsymbol{x}))(\boldsymbol{u}_T(\boldsymbol{x}+\boldsymbol{r}) \times \boldsymbol{u}_T(\boldsymbol{x}))\rangle = \tfrac{2}{15}hr^2,\ \ h = 2\nu\langle \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{\omega}\rangle, \qquad (1.1c)$$

where $\boldsymbol{r}$ is the separation vector between the two measurement points; the velocity component along $\boldsymbol{r}$ is called the longitudinal velocity $u_L = \boldsymbol{u}\cdot\hat{\boldsymbol{r}}$ (we will also refer to its vector representation $\boldsymbol{u}_L = u_L\hat{\boldsymbol{r}}$); the component orthogonal to $\boldsymbol{r}$ is called the transverse velocity $\boldsymbol{u}_T = \boldsymbol{u} - u_L\hat{\boldsymbol{r}}$; θ is the passive scalar. The quantities ε, ε_θ and h are, respectively, the mean dissipation rates of kinetic energy, passive scalar variance and helicity, and are defined in terms of the viscosity ν and the thermal diffusion coefficient α. The kinetic energy and passive scalar variance have positive-definite dissipation rates, while helicity does not. The latter fact, however, does not in itself preclude a helicity inertial range (Chen, Chen & Eyink 2003; Kurien, Taylor & Matsumoto 2004). Equations (1.1) are three of the few exact, non-trivial results known in the theory of statistical hydrodynamics. They are valuable benchmarks for the study of high-Reynolds-number (*Re*) turbulence in both experiments (Dhruva, Tsuji & Sreenivasan 1997; Mydlarski & Warhaft 1998; Chambers & Antonia 1984) and numerical simulations (Sreenivasan *et al.* 1996; Gotoh, Fukayama & Nakano 2002; Taylor, Kurien & Eyink 2003; Kurien *et al.* 2004).

In § 2 we derive the Kármán–Howarth equation for the two-point correlation function of q starting from the Boussinesq equations for rotating and stratified flows. As noted by Herring *et al.* (1994) for zero rotation and stratification, the flux and viscous-diffusion terms are not necessarily separated in scale (or wavenumber). However, in statistically homogeneous flow with $Re \to \infty$ and fixed Prandtl number Pr, there exists the possibility of a range of scales wherein the viscous-diffusion rate is at least sub-dominant, if not negligible, compared to the transfer rate. Assuming such a range of scales exists at sufficiently high Re, we recover a balance between the divergence of the third-order correlation between velocity and potential vorticity, and the production–dissipation rate of potential enstrophy.

In § 3 we consider different limits of the Rossby (Ro) and Froude (Fr) numbers. When rotation and stratification are small (Ro and Fr are large), we can reasonably hypothesize local isotropy at small-enough scales. Then, an exact scaling law analogous to (1.1) arises for the third-order mixed correlation of $\boldsymbol{u}$ and q, with a prefactor of 2/3. In the QG limit (Ro and Fr are small), the dependence on Ro and Fr may be scaled out in a coordinate system with vertical stretching. For QG scales in the stretched coordinates, we again assume local isotropy and deduce the same linear scaling law for the mixed third-order correlation. In § 4, we tabulate several intermediate limits in Ro and Fr; in some cases the correlation dynamics allow for a scale separation between flux terms and viscous-diffusion terms, and hence the clear possibility for an inertial range of scales dominated by potential enstrophy flux.

2. The Kármán–Howarth equation for potential vorticity q

The Boussinesq equations for rotating, stably stratified and incompressible flow are

$$\frac{\mathrm{D}}{\mathrm{D}t}\boldsymbol{u} + f\hat{\boldsymbol{z}} \times \boldsymbol{u} + \frac{1}{\rho_0}\nabla p + \frac{\tilde{\rho}}{\rho_0} g\hat{\boldsymbol{z}} = \nu\nabla^2\boldsymbol{u}, \quad \nabla\cdot\boldsymbol{u} = 0, \quad \frac{\mathrm{D}}{\mathrm{D}t}\tilde{\rho} - bw = \kappa\nabla^2\tilde{\rho}, \qquad (2.1)$$

where $\mathrm{D}/\mathrm{D}t = \partial_t + (\boldsymbol{u}\cdot\nabla)$, $\boldsymbol{u}$ is the velocity, w is its vertical component, p is an effective pressure, $f = 2\Omega$ is the Coriolis parameter, Ω is the constant background rotation rate, $\nu = \mu/\rho_0$ is the kinematic viscosity and κ is the mass diffusivity coefficient. The total density is $\rho(\boldsymbol{x}) = \rho_0 - bz + \tilde{\rho}(\boldsymbol{x})$, where ρ_0 is the constant background, b is also constant and larger than zero for stable stratification, and $\tilde{\rho}$ is the density fluctuation such that $|\tilde{\rho}| \ll |bz| \ll \rho_0$.

We begin with the equation for q in rotating stably stratified flow (Embid & Majda 1998) with periodic boundary conditions:

$$\frac{\mathrm{D}q}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{\omega}_a\cdot\nabla\rho) = \nu\nabla^2\boldsymbol{\omega}\cdot\nabla\rho + \kappa\nabla^2\nabla\tilde{\rho}\cdot\boldsymbol{\omega}_a \qquad (2.2)$$

where $\boldsymbol{\omega}_a = \boldsymbol{\omega} + f\hat{\boldsymbol{z}}$ is the absolute vorticity and $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$ is the relative vorticity. The equation for the potential enstrophy $Q = q^2/2$ is obtained by multiplying (2.2) by q. The mean production–dissipation rate of Q over the domain is then given by

$$\frac{\partial}{\partial t}\langle Q\rangle = \nu\langle q\nabla^2\boldsymbol{\omega}\cdot\nabla\rho\rangle + \kappa\langle q\nabla^2\nabla\tilde{\rho}\cdot\boldsymbol{\omega}_a\rangle = -\varepsilon_Q \qquad (2.3)$$

where $\langle\cdot\rangle$ denotes volume integration over the periodic domain. The quantity ε_Q is not sign-definite, allowing for both production and dissipation, of potential enstrophy (see also Herring *et al.* 1994).

2.1. *Equation for the two-point correlation function of q in homogeneous flow*

Following Embid & Majda (1998), one may define non-dimensional parameters:

$$Ro = \frac{U}{Lf}, \quad Fr = \frac{U}{LN}, \quad \Gamma = \frac{BgL}{U^2}, \quad Pr = \frac{\nu}{\kappa}, \quad \Lambda = \Gamma Fr^2, \quad Re = \frac{UL}{\nu}, \tag{2.4}$$

where U, L, L/U are the characteristic velocity, length scale and time scale, and $B\rho_0$ is the characteristic scale for the density fluctuations (B is a dimensionless constant). The buoyancy frequency N is given by $N = (gb/\rho_0)^{1/2}$. For the limiting cases $Fr \to 0$ and $Fr \to \infty$, conservation of total energy $(\boldsymbol{u}\cdot\boldsymbol{u} + \tilde{\rho}^2)/2$ requires that $\Gamma = 1/Fr$, and thus $\Lambda = Fr$. The latter equality is assumed throughout this work. Then the characteristic velocity is given by $U = Bg/N = B(\rho_0 g/b)^{1/2}$.

With the above definitions, the non-dimensional form of the equation for q as measured at point $\boldsymbol{x}$ becomes

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\boldsymbol{\omega}\cdot\nabla\tilde{\rho} + Ro^{-1}\frac{\partial\tilde{\rho}}{\partial z} - Fr^{-1}\omega_3\right] = Re^{-1}(Fr^{-1}\nabla^2\omega_3 - \nabla\tilde{\rho}\cdot\nabla^2\boldsymbol{\omega}) - (RePr)^{-1}\left(Ro^{-1}\nabla^2\frac{\partial\tilde{\rho}}{\partial z} + \boldsymbol{\omega}\cdot\nabla^2\nabla\tilde{\rho}\right) \tag{2.5}$$

where all variables are non-dimensional and ω_3 is the vertical component of $\boldsymbol{\omega}$. From now on we include in q only the contributions including the fluctuations $\boldsymbol{\omega}$ and $\tilde{\rho}$

$$q = \boldsymbol{\omega}\cdot\nabla\tilde{\rho} + Ro^{-1}\frac{\partial\tilde{\rho}}{\partial z} - Fr^{-1}\omega_3, \tag{2.6}$$

and we work entirely in the non-dimensional units.

One may write down an equation identical to (2.5) for q', the potential vorticity at point $\boldsymbol{x}'$. Cross-multiplication and summing of the two resulting equations yields the equation for the two-point quantity $qq' = q(\boldsymbol{x})q(\boldsymbol{x}')$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(qq') &= \frac{\mathrm{d}}{\mathrm{d}t}\Bigg[(\omega_i\partial_i\tilde{\rho})(\omega'_j\partial_{j'}\tilde{\rho}') + Ro^{-1}\left\{\frac{\partial\tilde{\rho}}{\partial z}\omega'_i\partial_{i'}\tilde{\rho}' + \frac{\partial\tilde{\rho}'}{\partial z'}\omega_i\partial_i\tilde{\rho}\right\} - Fr^{-1}\{\omega_3\omega'_i\partial_{i'}\tilde{\rho}' \\
&\quad + \omega'_3\omega_i\partial_i\tilde{\rho}\} + Ro^{-2}\frac{\partial\tilde{\rho}}{\partial z}\frac{\partial\tilde{\rho}'}{\partial z'} - Ro^{-1}Fr^{-1}\left\{\omega_3\frac{\partial\tilde{\rho}'}{\partial z'} + \omega'_3\frac{\partial\tilde{\rho}}{\partial z}\right\} + Fr^{-2}\omega_3\omega'_3\Bigg] \\
&= Re^{-1}\Bigg[(\omega_k\partial_k\tilde{\rho})(\partial^2_{j'}\omega'_i)(\partial_{i'}\tilde{\rho}') + (\omega'_k\partial_{k'}\tilde{\rho}')(\partial^2_j\omega_i)(\partial_i\tilde{\rho}) \\
&\quad + Ro^{-1}\left\{\frac{\partial\tilde{\rho}}{\partial z}(\partial^2_{j'}\omega'_i)(\partial_{i'}\tilde{\rho}') + \frac{\partial\tilde{\rho}'}{\partial z'}(\partial^2_j\omega_i)(\partial_i\tilde{\rho})\right\} \\
&\quad - Fr^{-1}\left\{\omega_3(\partial^2_{j'}\omega'_i)(\partial_{i'}\tilde{\rho}') + \omega'_3(\partial^2_j\omega_i)(\partial_i\tilde{\rho}) + (\omega_i\partial_i\tilde{\rho})(\partial^2_{j'}\omega'_3) + (\omega'_i\partial_{i'}\tilde{\rho}')(\partial^2_j\omega_3)\right\} \\
&\quad - Ro^{-1}Fr^{-1}\left\{\frac{\partial\tilde{\rho}}{\partial z}\partial^2_{j'}\omega'_3 + \frac{\partial\tilde{\rho}'}{\partial z'}\partial^2_j\omega_3\right\} + Fr^{-2}\{\omega_3\partial^2_{j'}\omega'_3 + \omega'_3\partial^2_j\omega_3\}\Bigg] \\
&\quad + Re^{-1}Pr^{-1}\Bigg[(\omega_k\partial_k\tilde{\rho})\omega'_i(\partial_{i'}\partial^2_{j'}\tilde{\rho}') + (\omega'_k\partial_{k'}\tilde{\rho}')\omega_i(\partial_i\partial^2_j\tilde{\rho}) \\
&\quad + Ro^{-1}\left\{\frac{\partial\tilde{\rho}}{\partial z}\omega'_i(\partial_{i'}\partial^2_{j'}\tilde{\rho}') + \frac{\partial\tilde{\rho}'}{\partial z'}\omega_i(\partial_i\partial^2_j\tilde{\rho}) + (\omega_i\partial_i\tilde{\rho})\left(\partial^2_{j'}\frac{\partial\tilde{\rho}'}{\partial z'}\right)\right. \\
&\quad \left. + (\omega'_i\partial_{i'}\tilde{\rho}')\left(\partial^2_j\frac{\partial\tilde{\rho}}{\partial z}\right)\right\} - Fr^{-1}\{\omega_3\omega'_i(\partial_{i'}\partial^2_{j'}\tilde{\rho}') + \omega'_3\omega_i(\partial_i\partial^2_j\tilde{\rho})\} \\
&\quad + Ro^{-2}\left\{\frac{\partial\tilde{\rho}}{\partial z}\partial^2_{j'}\frac{\partial\tilde{\rho}'}{\partial z'} + \frac{\partial\tilde{\rho}'}{\partial z'}\partial^2_j\frac{\partial\tilde{\rho}}{\partial z}\right\} - Ro^{-1}Fr^{-1}\left\{\omega_3\partial^2_{j'}\frac{\partial\tilde{\rho}'}{\partial z'} + \omega'_3\partial^2_j\frac{\partial\tilde{\rho}}{\partial z}\right\}\Bigg]
\end{aligned} \tag{2.7}$$

where $\mathrm{d}(qq')/\mathrm{d}t$ is defined by $\mathrm{d}(qq')/\mathrm{d}t = \partial_t(qq') + q'u_i\partial_i q + qu_i'\partial_{i'}q'$. Here all primed variables denote their values at $\boldsymbol{x}'$ and $\partial_{i'}$ denotes differentiation with respect to $\boldsymbol{x_i}'$.

Next one may express the equation in terms of the separation vector $\boldsymbol{r}$ between the points $\boldsymbol{x}$ and $\boldsymbol{x}'$, and rewrite it in terms of the new independent variables $\boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x}$ and $\boldsymbol{X} = (\boldsymbol{x} + \boldsymbol{x}')/2$ (see for example Hill 2002), where

$$\partial_i = -\partial_{r_i} + \tfrac{1}{2}\partial_{X_i}, \quad \partial_{i'} = \partial_{r_i} + \tfrac{1}{2}\partial_{X_i}, \quad \partial_i^2 = \partial_i'^2 = \partial_{r_i}^2 + \tfrac{1}{4}\partial_{X_i}^2. \tag{2.8}$$

The procedure to derive the equation for the two-point correlation function is familar from, for example, Frisch (1995) and Hill (2002). One can perform a change of variables and then ensemble average the equation assuming statistical homogeneity. Ensemble averaging commutes with the derivative operations ∂_{r_i} and ∂_{X_i}. Statistical homogeneity, as in the case of periodic boundary conditions with constant N and f, implies that the derivative operation ∂_{X_i} acting on any statistic yields zero. The result is

$$\frac{\partial\langle q\,q'\rangle}{\partial t} - \partial_{r_i}\langle q\,q'(u_i - u_i')\rangle = Re^{-1}\partial_{r_i}\left\langle q\rho'\partial_{j'}^2\omega_i' - q'\rho\partial_j^2\omega_i\right\rangle + (RePr)^{-1}\partial_{r_i}\left\langle q\omega_{a_i}'\partial_{j'}^2\tilde{\rho}' - q'\omega_{a_i}\partial_j^2\tilde{\rho}\right\rangle, \tag{2.9}$$

where all factors of Ro and Fr are 'hidden' in the non-dimensional expressions for q and q' defined by (2.6), $\rho = -Fr^{-1}z + \tilde{\rho}$ and $\boldsymbol{\omega}_a = Ro^{-1}\hat{\boldsymbol{z}} + \boldsymbol{\omega}$. Relation (2.9) is the non-dimensional Kármán–Howarth equation for the two-point second-order correlation function of q. It describes the dynamics of the two-point correlation of q in a statistically homogeneous Boussinesq flow, and is the starting point to consider various limiting cases as described below.

2.1.1. *Non-diffusive small-scale limit in homogeneous flow*

For finite Re and Pr, the viscous-diffusion terms on the right-hand side of (2.9) may not, in general, act only at scales much smaller than the nonlinear (transfer) terms on the left-hand side. Furthermore, each individual term on the right-hand side is not sign-definite, and thus allows for both production and dissipation of $\langle q\,q'\rangle$ at a given scale. This is reminiscent of the sign-indefiniteness of the dissipation rate of helicity and helical velocity statistics; despite this, it has recently been shown that an inertial range of helicity can exist with an exact scaling law recovered theoretically by Chkhetiani (1996) (see also L'vov *et al.* 1997; Gomez *et al.* 2000; Kurien 2003), as well as in simulations by Kurien *et al.* (2004).

We proceed with a particular order of limits, first $Re \to \infty$ with Pr fixed, followed by $r \to 0$. The expectation is that this procedure will access a range of (sufficiently small) scales such that the effect of the diffusion is sub-leading compared to the transfer for a given scale. One may then derive the leading-order balance between the flux and production-dissipation terms. In the first limit, the right-hand side of (2.9) becomes negligible for given r. Then, in the limit as $r \to 0$, $\partial_t\langle qq'\rangle \to 2\partial_t\langle Q\rangle = -2\varepsilon_Q$ and hence (2.9) reduces to

$$\nabla_{\boldsymbol{r}} \cdot \langle q\,q'(\boldsymbol{u} - \boldsymbol{u}')\rangle = -2\varepsilon_Q. \tag{2.10}$$

Implicit in this step is the assumption that the production-dissipation rate ε_Q remains finite in the non-diffusive limit $Re \to \infty$, Pr fixed. This is analogous to the Kolmogorov (1941) hypothesis that kinetic energy dissipation rate remains finite in the inviscid limit. Similar assumptions were used for the rates of dissipation of passive scalar variance and helicity to derive (1.1). Equation (2.10) is analogous to the result for third-order velocity structure functions in statistically homogeneous, high-Reynolds-number turbulence in the small-scale limit (Frisch 1995).

3. Limiting cases in the rotation and stratification parameters

Our starting point is the Kármán–Howarth equation for homogeneous flows, equation (2.9), upon which we impose limits in *Ro* and *Fr* using (2.6) (see also (2.7)). In the large-*Ro* large-*Fr* limit, we derive an exact balance for the isotropic small scales between ε_Q and the mixed third-order correlation of $\boldsymbol{u}$ and q. The scaling law thus derived, which we will call the '2/3-law', is analogous to the established scaling laws for kinetic energy, passive scalar variance and helicity in (1.1). The same 2/3-law is shown to hold for the QG limit in a stretched coordinate system, presumably in a range of larger scales for which vertical stretching leads to local isotropy.

3.1. *Rotation and stratification are small (large Ro, large Fr)*

The equations for large *Ro* and large *Fr* should tend toward the equations for incompressible variable-density three-dimensional flow. Consider the following scalings:

$$Ro = \frac{1}{\epsilon}\frac{N}{f}, \quad Fr = \frac{1}{\epsilon}, \quad Re \geqslant O(1), \quad Pr \ \text{fixed as } \epsilon \to 0. \tag{3.1}$$

After the change of variables according to (2.8), ensemble averaging and assuming homogeneity, the leading-order $O(1)$ balance of (2.9) is

$$\frac{\partial \langle q\, q'\rangle}{\partial t} - \partial_{r_i}\langle q\, q'(u_i - u_i')\rangle = Re^{-1}\left[\partial_{r_i}\langle q\, \tilde{\rho}'\nabla'^2\omega_i'\rangle - \partial_{r_i}\langle q'\tilde{\rho}\nabla^2\omega_i\rangle\right]$$
$$+ Re^{-1}Pr^{-1}\left[\partial_{r_i}\langle q\, \omega_i'\nabla'^2\tilde{\rho}'\rangle - \partial_{r_i}\langle q'\omega_i\nabla^2\tilde{\rho}\rangle\right], \quad \text{where } q \sim \boldsymbol{\omega}\cdot\nabla\tilde{\rho}. \tag{3.2}$$

Note that, unlike in the equation for velocity correlations, the order of the derivative with respect to $\boldsymbol{r}$ is the same for the flux and viscous-diffusion terms. Thus we cannot, without further assumptions, expect an 'inertial-range' transfer dominated by the flux of potential enstrophy in some range of scales. In this limit, the equations of motion are identical to those investigated by Herring *et al.* (1994), that is, the density is a passive scalar and the momentum obeys the Navier–Stokes equations.

3.1.1. *Local isotropy*

In the large-*Ro* and large-*Fr* limit, it is reasonable to further assume local isotropy for sufficiently small scales, that is, invariance of the correlation tensors under arbitrary rigid rotations. Invariance with rotation by π radians about each of the coordinate axes yields the constraint that only the longitudinal components of the tensor correlations are non-zero. Then (3.2) reduces to

$$\frac{\partial\,\langle q\, q'\rangle}{\partial t} - \nabla_{\boldsymbol{r}}\cdot\langle q\, q'(\boldsymbol{u}_L - \boldsymbol{u}_L')\rangle = Re^{-1}\nabla_{\boldsymbol{r}}\cdot\left\langle q\rho'\partial_{j'}^2\boldsymbol{\omega}_L' - q'\rho\partial_j^2\boldsymbol{\omega}_L\right\rangle$$
$$+ (Re\, Pr)^{-1}\nabla_{\boldsymbol{r}}\cdot\left\langle q\boldsymbol{\omega}_{aL}'\partial_{j'}^2\tilde{\rho}' - q'\boldsymbol{\omega}_{aL}\partial_j^2\tilde{\rho}\right\rangle, \tag{3.3}$$

where subscript L denotes the longitudinal component. Equation (3.3) is a Kármań–Howarth equation for two-point potential vorticity statistics in three-dimensional incompressible, variable-density, statistically homogeneous and locally isotropic flow. We may use the following isotropic forms for the scalar and tensor correlations in (3.3):

$$\left.\begin{aligned}\langle q\,q'\rangle = \langle q(\boldsymbol{x})q(\boldsymbol{x}+\boldsymbol{r})\rangle = C(r), \quad \langle q(\boldsymbol{x})q(\boldsymbol{x}+\boldsymbol{r})(u_i(\boldsymbol{x})-u_i(\boldsymbol{x}+\boldsymbol{r}))\rangle = F(r)\frac{r_i}{r},\\ \left\langle q\rho'\partial^2_{j'}\omega'_i - q'\rho\partial^2_j\omega_i\right\rangle = G_1(r)\frac{r_i}{r}, \quad \left\langle q\omega'_{a_i}\partial^2_{j'}\tilde{\rho}' - q'\omega_{a_i}\partial^2_j\tilde{\rho}\right\rangle = G_2(r)\frac{r_i}{r},\end{aligned}\right\} \tag{3.4}$$

where $C(r)$, $F(r)$, $G_1(r)$ and $G_2(r)$ are scalar functions of r. Substituting these isotropic forms into (3.3) and using the identities

$$\frac{\partial r}{\partial r_i} = \frac{r_i}{r}, \quad \frac{\partial}{\partial r_i} = \frac{\partial r}{\partial r_i}\frac{\partial}{\partial r} = \frac{r_i}{r}\frac{\partial}{\partial r} \tag{3.5}$$

we find

$$\frac{\partial C(r)}{\partial t} - \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F(r)) = Re^{-1}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2G_1(r)) + (Re\,Pr)^{-1}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2G_2(r)). \tag{3.6}$$

Relation (3.6) is the the scalar form of the Kármán–Howarth equation for the second-order moment $\langle qq'\rangle$ in locally isotropic flow.

3.1.2. *Statistically steady state, non-diffusive and small-scale limit*

The analogy to the Kolmogorov 4/5-law for longitudinal structure functions may be recovered in the non-diffusive limit using the following steps. First, assume a statistically steady state in time. Then, in order to observe the limit where the viscosity and mass diffusion contributions are small, take the limit $Re \to \infty$ with Pr constant, eliminating the right-hand side of (3.6). Follow this with $r \to 0$ to obtain

$$\frac{\partial}{\partial t}C(r)|_{r=0} + \frac{\partial}{\partial t}\left(\left.\frac{\partial C(r)}{\partial r}\right|_{r=0} r\right) + \cdots + \left.\frac{1}{r^2}\frac{\partial}{\partial r}(r^2F(r))\right|_{r\to 0} = 0. \tag{3.7}$$

The mean dissipation rate of the potential enstrophy is the leading-order term of the time derivative:

$$\frac{\partial}{\partial t}(C(0)) = \frac{\partial}{\partial t}\langle q^2\rangle = 2\frac{\partial}{\partial t}\langle Q\rangle = -2\varepsilon_Q \tag{3.8}$$

where it is assumed that the higher-order contributions to the time derivative in (3.6) vanish as $r \to 0$ in the statistically steady state. This is analogous to the assumption made to derive the Kármán–Howarth equation for helicity (Kurien 2003). Next use (3.7)–(3.8) in (3.6), multiply by r^2 throughout, and integrate with respect to r to find

$$F(r) \sim -\tfrac{2}{3}\varepsilon_Q r, \quad r \to 0. \tag{3.9}$$

The constant of integration is zero assuming that the scalar function $F(r)$ remains regular as $r \to 0$. Alternatively, we can write the '2/3-law' for the third-order correlation of q and velocity as

$$\langle q(\boldsymbol{x})q(\boldsymbol{x}+\boldsymbol{r})(\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}))\cdot\hat{\boldsymbol{r}})\rangle = -\tfrac{2}{3}\varepsilon_Q r \tag{3.10}$$

where ε_Q is obtained from (2.3) in the large-*Ro*, large-*Fr* limit,

$$\varepsilon_Q = -(\nu\langle q\nabla^2\boldsymbol{\omega}\cdot\nabla\tilde{\rho}\rangle + \kappa\langle q\nabla^2\nabla\tilde{\rho}\cdot\boldsymbol{\omega}\rangle). \tag{3.11}$$

Equation (3.10) is the potential enstrophy counterpart to the scaling laws presented in (1.1). As for the helicity case, ε_Q is not sign definite. As we have discussed above, this fact in itself does not exclude the possibility of an inertial range.

3.2. *The quasi-geostrophic limit*

The quasi-geostrophic (QG) limit is attained in the high-rotation, large-stratification (low-Ro, low-Fr) limit corresponding to

$$Ro = \epsilon \frac{N}{f}, \quad Fr = \epsilon, \quad Re \geqslant O(1), \quad Pr \text{ fixed as } \epsilon \to 0. \tag{3.12}$$

After change of variables, ensemble averaging and using homogeneity, the leading-order $O(1/\epsilon^2)$ terms from (2.7) are

$$\left.\begin{aligned} &\frac{\partial\langle q\,q'\rangle}{\partial t} - \partial_{r_i}\langle q\,q'(u_i - u_i')\rangle = Re^{-1}\left(2\partial^2_{r_j}\langle\omega_3\omega_3'\rangle - \frac{f}{N}\left\{\partial^2_{r_j}\left\langle\frac{\partial\tilde{\rho}}{\partial z}\omega_3'\right\rangle + \partial^2_{r_j}\left\langle\frac{\partial\tilde{\rho}'}{\partial z'}\omega_3\right\rangle\right\}\right)\\ &\qquad + Re^{-1}Pr^{-1}\left(\frac{f^2}{N^2}\partial^2_{r_j}\left\langle\frac{\partial\tilde{\rho}}{\partial z}\frac{\partial\tilde{\rho}'}{\partial z'}\right\rangle - \frac{f}{N}\left\{\partial^2_{r_j}\left\langle\omega_3\frac{\partial\tilde{\rho}'}{\partial z'}\right\rangle + \partial^2_{r_j}\left\langle\omega_3'\frac{\partial\tilde{\rho}}{\partial z}\right\rangle\right\}\right),\\ &q \sim \omega_3 - \frac{f}{N}\frac{\partial\tilde{\rho}}{\partial z}. \end{aligned}\right\} \tag{3.13}$$

In this limit, the viscous-diffusion terms on the right-hand side of (3.13) are all second-order derivatives with respect to $\boldsymbol{r}$. Therefore they can be considered localized at small scales, allowing an inertial range in which viscous-diffusion contributions are separated in scale from transfer contributions. To focus on QG scales rather than small scales, we scale out the dependence on f and N by using a stretched z-coordinate, $z_* = (N/f)z$ (Charney 1971). Then the scaled q is given by $q = \omega_3 - \partial\tilde{\rho}/\partial z_*$, and all z-derivatives $(f/N)\partial/\partial_z$ in (3.13) become $\partial/\partial z_*$. Such vertical stretching removes anisotropy in the QG scales; one may imagine pancake-shaped eddies, stretched to become spherical eddies. Following §§ 3.1.1 and 3.1.2, one may next derive the Kármán–Howarth equation analogous to (3.6), and then a scaling law for the third-order mixed correlation function in an isotropic range of scales, identical to (3.10). However, the separation distance r is measured in the stretched coordinates, and the mean dissipation rate for QG is given by

$$\varepsilon_Q = -\left[Re^{-1}\left\langle\frac{\partial\tilde{\rho}}{\partial z}\nabla^2_*\omega_3\right\rangle + (RePr)^{-1}\left\langle\omega_3\nabla^2_*\frac{\partial\tilde{\rho}}{\partial z}\right\rangle\right], \quad \nabla^2_* = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{N^2}{f^2}\frac{\partial^2}{\partial z_*^2}.$$

4. Summary and discussion

Our results for various limits of Ro and Fr are summarized in table 1. In cases (ii–iv), an inertial range of scales dominated by flux of potential enstrophy may be realized at finite Re because the viscous-diffusion terms are confined to small scales. For QG flow with $Ro, Fr \to 0$ (case ii) and $Re \to \infty$, the 2/3-law describes locally isotropic flow in a coordinate system with vertical stretching. In the remaining three cases (i, v and vi), the possibility of a potential-enstrophy inertial range at finite Re is not as clear because the flux and viscous-diffusion contributions can in principle intermingle at all scales. Nevertheless, the 2/3-law, equation (3.10), is obtained for the $Ro, Fr \to \infty$ case (i), again for locally isotropic flow in the $Re \to \infty$ limit. In the derivation of the 2/3-law for both cases (i) and (ii), there is a prescribed sequence of limits: $Re \to \infty$ with Pr fixed, followed by the separation $r \to 0$. In case (i), the 2/3-law is additional statistical information describing three-dimensional turbulent flow with a passive scalar $\tilde{\rho}$. A point of difference from (1.1*b*) is that the 2/3-law for q contains information about the geometry and structures of the flow; recall that $\boldsymbol{\omega}$

Case	*Ro*	*Fr*	q	Viscous-diffusion terms
i	$\frac{1}{\epsilon}\frac{N}{f}$	$\frac{1}{\epsilon}$	$\boldsymbol{\omega}\cdot\nabla\tilde{\rho}$	$Re^{-1}\partial_{r_i}\left\langle\left(q\tilde{\rho}'\partial^2_{j'}\omega'_i - q'\tilde{\rho}\partial^2_j\omega_i\right)\right\rangle$ $+(RePr)^{-1}\partial_{r_i}\left\langle\left(q\omega'_i\partial^2_{j'}\tilde{\rho}' - q'\omega_i\partial^2_j\tilde{\rho}\right)\right\rangle$
ii	$\epsilon\frac{N}{f}$	ϵ	$\omega_3 - \frac{f}{N}\frac{\partial\tilde{\rho}}{\partial z}$	$Re^{-1}\partial^2_{r_j}\left\langle 2(\omega_3\omega'_3) - \frac{f}{N}(\frac{\partial\tilde{\rho}}{\partial z}\omega'_3 + \frac{\partial\tilde{\rho}'}{\partial z'}\omega_3)\right\rangle$ $+(RePr)^{-1}\partial^2_{r_j}\left\langle\frac{f^2}{N^2}\frac{\partial\tilde{\rho}}{\partial z}\frac{\partial\tilde{\rho}'}{\partial z'} - \frac{f}{N}\left(\omega_3\frac{\partial\tilde{\rho}'}{\partial z'} + \omega'_3\frac{\partial\tilde{\rho}}{\partial z}\right)\right\rangle$
iii	ϵ	$O(1)$	$\frac{\partial\tilde{\rho}}{\partial z}$	$2(RePr)^{-1}\partial^2_{r_j}\langle qq'\rangle$
iv	$O(1)$	ϵ	ω_3	$2Re^{-1}\partial^2_{r_j}\langle qq'\rangle$
v	$\frac{1}{\epsilon}$	$O(1)$	$Fr^{-1}\omega_3 + \omega_i\partial_i\tilde{\rho}$	$Re^{-1}\partial_{r_i}\left\langle\left(q\rho'\partial^2_{j'}\omega'_i - q'\rho\partial^2_j\omega_i\right)\right\rangle$ $+(RePr)^{-1}\partial_{r_i}\langle(q\omega'_i\partial^2_{j'}\tilde{\rho}' - q'\omega_i\partial^2_j\tilde{\rho})\rangle$
vi	$O(1)$	$\frac{1}{\epsilon}$	$Ro^{-1}\frac{\partial\tilde{\rho}}{\partial z} + \omega_i\partial_i\tilde{\rho}$	$Re^{-1}\partial_{r_i}\left\langle\left(q\tilde{\rho}'\partial^2_{j'}\omega'_i - q'\tilde{\rho}\partial^2_j\omega_i\right)\right\rangle$ $+(RePr)^{-1}\partial_{r_i}\left\langle\left(q\omega'_{a_i}\partial^2_{j'}\tilde{\rho}' - q'\omega_{a_i}\partial^2_j\tilde{\rho}\right)\right\rangle$

TABLE 1. The form of q and the viscous-diffusion terms of (2.9) in various cases.

evolves as a line element and $\nabla\tilde{\rho}$ evolves as a surface element, and thus that q evolves as a volume element (Ertel 1942).

The 2/3-laws are derived assuming local isotropy for some range of scales: 'sufficiently small' scales for nearly isotropic flow with $Ro, Fr \to \infty$; larger QG scales for which vertical stretching removes anisotropy in the case of $Ro, Fr \to 0$. It is important to note that even if local isotropy is not strictly realized, there is always an isotropic component of the flow that could obey the 2/3-law (see Taylor *et al.* 2003). We expect the 2/3-laws to become part of the theoretical foundation of rotating and stratified flows, and a benchmark for high-Reynolds-number physical and numerical experiments, just as (1.1) provide a foundation for isotropic turbulence research.

An eventual goal is to elucidate the connection between energy and potential enstrophy in different parameter limits, as pioneered by Charney (1971) for QG dynamics. It is instructive to write the conservation laws for energy and potential enstrophy:

$$\frac{\partial}{\partial t}\langle|\boldsymbol{v}|^2 + \tilde{\rho}^2\rangle = Re^{-1}\langle\nabla^2|\boldsymbol{v}|^2\rangle + (RePr)^{-1}\langle\nabla^2\tilde{\rho}^2\rangle,$$

$$\frac{\partial\langle Q\rangle}{\partial t} = Re^{-1}\left((Fr^{-1}\langle q\nabla^2\omega_3\rangle - \langle q\nabla\tilde{\rho}\cdot\nabla^2\boldsymbol{\omega}\rangle) - Pr^{-1}\left(Ro^{-1}\left\langle q\nabla^2\frac{\partial\tilde{\rho}}{\partial z}\right\rangle + \langle q\,\boldsymbol{\omega}\cdot\nabla^2\nabla\tilde{\rho}\rangle\right)\right),$$

where $q = \boldsymbol{\omega}\cdot\nabla\tilde{\rho} + Ro^{-1}\partial\tilde{\rho}/\partial z - Fr^{-1}\omega_3$. The energy conservation law does not depend on either *Ro* or *Fr*, while the conservation law for potential enstrophy does (see also table 1). Thus potential enstrophy may constrain energy transfer among different scales in cases other than the QG limit, an exciting possibility for future research.

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